No, Really, How Airplanes Turn? – Do The Math (or Physics)

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The Problem

The internet is littered with the conventional simplistic aerodynamic wisdom on how airplanes turn. Some examples from NASA and others who should know. The website quotes are provided in the appendix.

The essence of nearly all the stories is this: Aircraft banks, a horizontal component of lift results, this unopposed centripetal force causes the aircraft to rotate on a circular path about the center of the turn (and BTW where turn radius r is r= V^2/(g*sin(bank angle)) from Newton's 2^{nd} law assuming circular motion).

And even the FAA in its written test for pilots says the answer to the question in the title above is the horizontal component of lift.

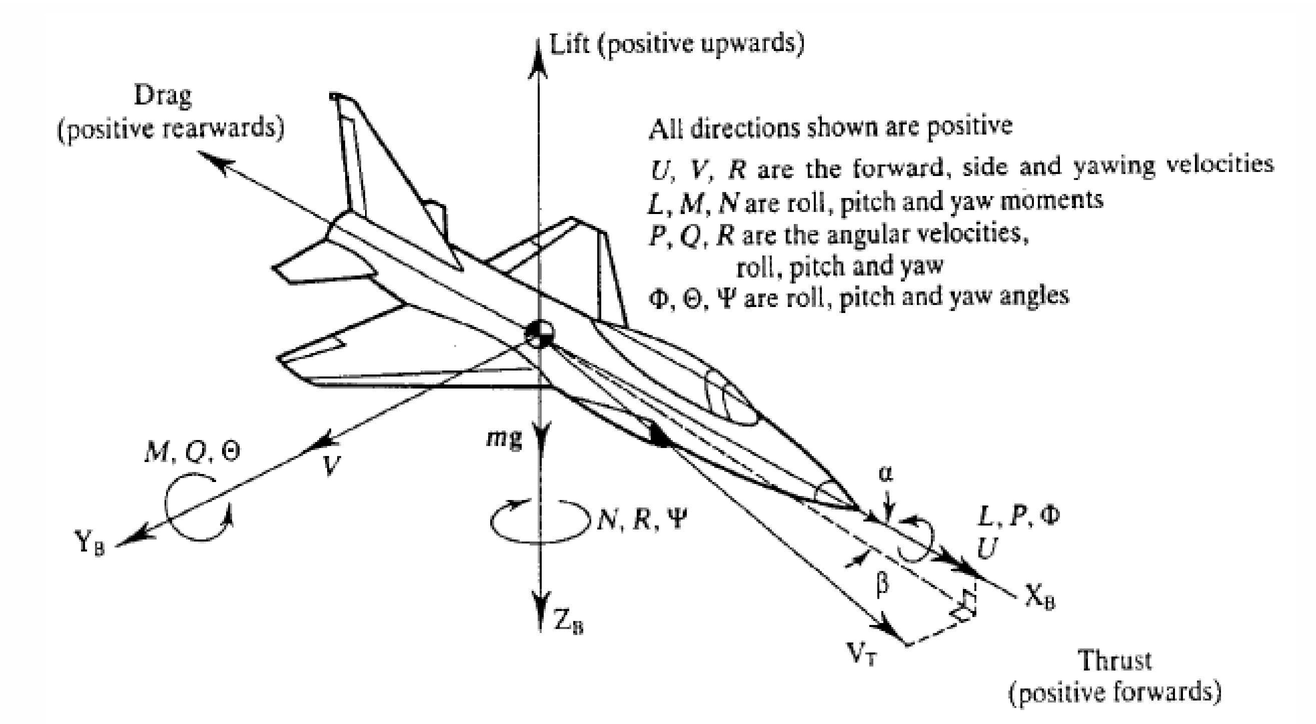
However, what seriously appears to be missing in the stories is just why the aircraft heading should be turning since there has been no mention of any torque about the yaw axis (axis nominally in the vertical direction) of the plane. Without this torque, the heading has not reason to change.

Source of a Torque (or "Moment")

First we have to suffer some pain to develop an understanding and notation for the mathematical representation of the solution of the equations of motion for aircraft. Figure 1 from Fossen [1] defines the relevant parameters, although other sources are available [2]. Before getting into that, note that in the figure there is shown an angle β that is the angle between the "relative wind" and the longitudinal axis of the plane (when the pitch angle α is zero). This is sometimes called the sideslip angle and it is nominally the angle between the longitudinal axis projection on the ground, or "heading," and the direction of motion of the center of mass projection on the ground, or "course," of the plane. The course is in the opposite direction of the relative wind, which is the apparent direction of the wind on the aircraft in the fixed body frame of reference.

When the airplane banks, a lift force in the horizontal direction becomes lift*sin(bank angle). This provides a lateral acceleration and thus an increasing lateral velocity, V. That leads to a finite sideslip angle β. This sideslip causes the relative wind to hit the vertical tail at that angle β which produces a lateral force on the tail due to the vertical tail angle of attack. The force is directed opposite to that from the horizontal lift component (but not equal). Since the force is not at the center of gravity (CG) of the aircraft, there is also a moment that results. We will shortly explore the effects of that force and moment.

Equations of Motion



But first we will look at the equations of motion without worrying about most of the aerodynamic forces that can appear. From Fossen the parameters are given here. U,V and W are the velocity components in the **BODY** frame of reference (XB,YB,ZB) above.

$$\begin{bmatrix} U \\ V \\ W \\ P \\ Q \\ R \end{bmatrix} = \begin{bmatrix} \text{longitudinal (forward) velocity} \\ \text{lateral (transverse) velocity} \\ \text{vertical velocity} \\ \text{roll rate} \\ \text{pitch rate} \\ \text{yaw rate} \end{bmatrix}$$

Here W is called the "vertical velocity" but this means in the direction perpendicular to XB (longitude) and YB (lateral), which is <u>not generally vertical in the (inertial) earth frame of reference</u>. The angular velocity "rates" are about the body axes.

The 6 Degrees Of Freedom (6DOF) coordinates in the fixed earth frame are defined as:

$$\begin{bmatrix} X_E \\ Y_E \\ Z_E, h \\ \Phi \\ \Theta \\ \Psi \end{bmatrix} = \begin{bmatrix} \text{Earth-fixed } x\text{-position} \\ \text{Earth-fixed } y\text{-position} \\ \text{Earth-fixed } z\text{-position (axis downwards), altitude} \\ \text{roll angle} \\ \text{pitch angle} \\ \text{yaw angle} \end{bmatrix}$$

The XE YE ZE here are Earth fixed and the 3 Euler angles provide the aircraft orientation in the earth frame. (In the body frame, the angles are zero by definition of course.)

In the (rather unfortunate) notation of Fossen, the forces and moments in the body frame are called:

$$\begin{bmatrix} X \\ Y \\ Z \\ L \\ M \\ N \end{bmatrix} := \begin{bmatrix} \text{longitudinal force} \\ \text{transverse force} \\ \text{vertical force} \\ \text{roll moment} \\ \text{pitch moment} \\ \text{yaw moment} \end{bmatrix}$$

Again "vertical force" is generally not "up." Note that the X, Y and Z here are not positions but forces which could perhaps better have been called FX FY and FZ. Next Newton's equations of motion, from **F**=m**a**, can be written out using the Euler angle coordinate transformation from earth to body (again XYZ are forces in body frame) giving:

$$\begin{aligned}
m(\dot{U} + QW - RV + g\sin(\Theta)) &= X \\
m(\dot{V} + UR - WP - g\cos(\Theta)\sin(\Phi)) &= Y \\
m(\dot{W} + VP - QU - g\cos(\Theta)\cos(\Phi)) &= Z \\
I_x\dot{P} - I_{xz}(\dot{R} + PQ) + (I_z - I_y)QR &= L \\
I_y\dot{Q} + I_{xz}(P^2 - R^2) + (I_x - I_z)PR &= M \\
I_z\dot{R} - I_{xz}\dot{P} + (I_y - I_x)PQ + I_{xz}QR &= N
\end{aligned}$$

Here the I's are moments of inertia (after using lateral symmetry) about the body axes. Typically Ixz is small and we might take it as zero for an airplane. To complete the equations and allow determination of the resulting earth trajectory and aircraft orientation we use matrix notation:

$$\begin{bmatrix} \dot{X}_E \\ \dot{Y}_E \\ \dot{Z}_E \end{bmatrix} = \begin{bmatrix} c\Psi c\Theta & -s\Psi c\Phi + c\Psi s\Theta s\Phi & s\Psi s\Phi + c\Psi c\Phi s\Theta \\ s\Psi c\Theta & c\Psi c\Phi + s\Phi s\Theta s\Psi & -c\Psi s\Phi + s\Theta s\Psi c\Phi \\ -s\Theta & c\Theta s\Phi & c\Theta c\Phi \end{bmatrix} \begin{bmatrix} U \\ V \\ W \end{bmatrix}$$

and

$$\begin{bmatrix} \dot{\Phi} \\ \dot{\Theta} \\ \dot{\Psi} \end{bmatrix} = \begin{bmatrix} 1 & s\Phi t\Theta & c\Phi t\Theta \\ 0 & c\Phi & -s\Phi \\ 0 & s\Phi / c\Theta & c\Phi / c\Theta \end{bmatrix} \begin{bmatrix} P \\ Q \\ R \end{bmatrix}$$

Here s, c and t are the obvious trig functions. The meaning, for example, from the lower equation, d(Φ)/dt is

$d(\Phi)/dt = P + \sin(\Phi)*\tan(\Theta)*Q + \cos(\Phi)*\tan(\Theta)*R$

after multiplying out the matrices.

All this seems intimidating, especially when you generally must provide forces and moments describing the highly complex and overlapping effects of all the aerodynamic interaction of the plane's surfaces, including the control surfaces and thrust.

Proposed Simplified Case for a Turning Study

The conventional story on turning, as indicated before, is basically aircraft banks and the turn follows largely automatically with maybe small control corrections to hold altitude. It is often emphasized that the rudder is not required. So let us simplify to an example for turning which has an aircraft in straight and level flight quickly banks and holds a constant roll angle Φ. The pitch angle Θ is taken as constant at zero and the yaw angle Ψ is initially zero. Any "vertical" body motion is ignored so W=0 and all motion will be in the XE, YE plane.

Thus far there are no moments (torques) but the roll and pitch are held constant so P=Q=0 and the lift is always in the -ZB direction with zero component in the YB (body) direction (even when banked). As a potential source of a moment, we will assume that if the sideslip angle β ceases to be zero, there will be a force Fv perpendicular to the vertical tail surface, related to its angle of attack from the relative wind. Generally Fv(β) is considered to be approximately linear in β. This force will be directed opposite to the y velocity V. The tail has a moment arm with respect to the CG, say 'at', so the yaw moment N will be Fv*at but still L=M=0 (at least if the vertical tail is centered on the longitudinal axis if you want to get picky). So we can write approximate equations as:

$dU/dt = VR$
$dV/dt = g*\sin(\Phi) - UR - Fv(\beta)/m$ Eqs. X
$dR/dt = Fv(\beta)*at /Iz$
$dW/dt=0$

$d(\Psi)/dt = R*\cos(\Phi)$
$d(XE)/dt = U*\cos(\Psi) - V*\sin(\Psi)*\cos(\Phi)$
$d(YE)/dt = U*\sin(\Psi) + V*\cos(\Psi)*\cos(\Phi)$

where $\beta = \text{atan}(V/U)$.

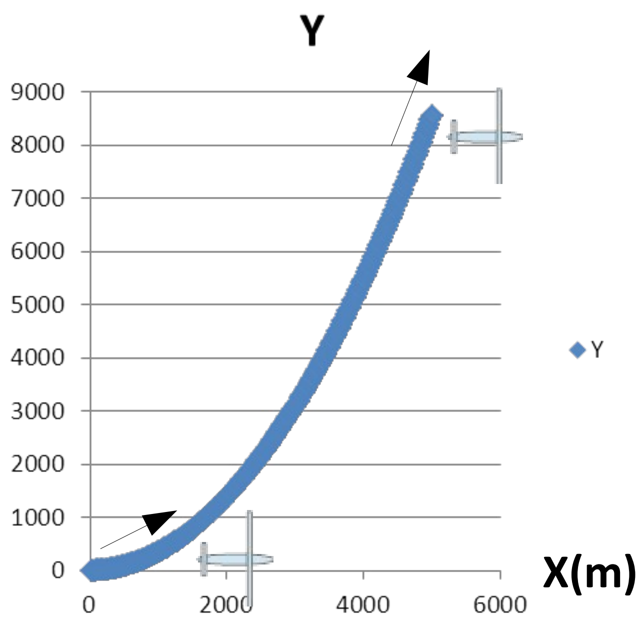
Solutions to the Simple Equations

A. First we ignore the potential force and moment from the vertical tail and remove Fv.
A.1 Analytic
By inspection we see that then $R=\Psi=0$ and the only relevant part remaining is

$dV/dt = g*\sin(\Phi)$ so
$d(XE)/dt = U$
$d(YE)/dt = V*\cos(\Phi)$

So U=constant and $V= t * g*\sin(\Phi)*\cos(\Phi)$ providing $XE=Uo*t$ and $YE=1/2*g*t^2*\sin(\Phi)*\cos(\Phi)$ which is a parabola in XY space with no change in heading (Ψ=0). Certainly not a turn. The trajectory found from the numerical integration of the Fv=0 equations in XEvsYE is provided below just to verify the inspection result. This is for a fixed roll angle of Φ =10 degrees. Of course the yaw angle Ψ remains at zero so the heading is always in the X direction.

A.2 Numerical
Below is the numerical trajectory when Fv is set to zero. As indicated by the analysis discussion, the result is a parabola with an ever more rapid velocity component in the YE direction. As indicated by the airplane graphic, the heading remains on the XE direction.



Parabolic trajectory over 100 seconds for Fv=0.

B. Next we return the Fv factors to provide a mechanism for a moment and additionally a force that can alter the y direction motion. For this we need some physical values for the aircraft. Taking a nominal Cessna 172, for round numbers we use g=10m/s/s, Uo=50 m/s, m=1000 kg, at=5 meters, z moment Iz ~ 3000 kg-m^2, air density ~1 kg/m^3, vertical tail area 1.65 m^2 and take a coefficient of lift CL of .5*(β/6°) so
Fv = 1/2*rho*U^2*Area*CL. For example, if β = 1°, this gives ~170 Newtons whereas, just for a scale comparison, the lift component in the horizontal direction for a 10 degree roll angle is about 1700 Newtons. This example value of Fv leads to an instantaneous dR/dt ~ 0.3 rad/s/s of yaw, which is very large, suggesting the solution value for β must be much smaller that 1 degree. So the vertical tail force and its moment have the potential for a significant impact on motion, depending critically on the dynamic value of β. Note that β=0 cannot be a solution giving turning motion since that is equivalent to removing Fv.

B1. Analytic
The more complicated equations with Fv do not immediately lent themselves to any obvious analytic solution. However if we assume that the equations can be linearized some progress can be made. If we Eq. X and use the U, V and R portion assuming they can can be written as U = Uo + U1 exp(-α+iω)t , etc. and substitute into the dV/dt

equation we find for a perturbation about circular motion

$g*\sin(\Phi) = UoRo$ at zeroth order. This is consistent with circular motion since $Ro=2\pi/\text{Period}$. For a constant speed Uo the radius would satisfy $2\pi r= Uo*\text{Period}$ so $r=U0^2/(g*\sin(\Phi)$.

The dU/dt at zeroth order then gives Vo=0 since Ro is not zero. So since Fv vanishes at zero β and $\beta \sim V1/U0$ is only first order, the d(R)/dt has no zeroth order parts.

If you press forward to the first order equations it becomes slightly more complicated and it is left to the student to verify these results when $\cos\Phi$ is taken as 1 for cleanliness.

$\alpha = dFv/dt(\beta=0)/(2mUo)$ and

$\omega = \text{sqrt}(\alpha^2 + Ro^2 + dFv/dt(\beta=0)*at/Iz)$.

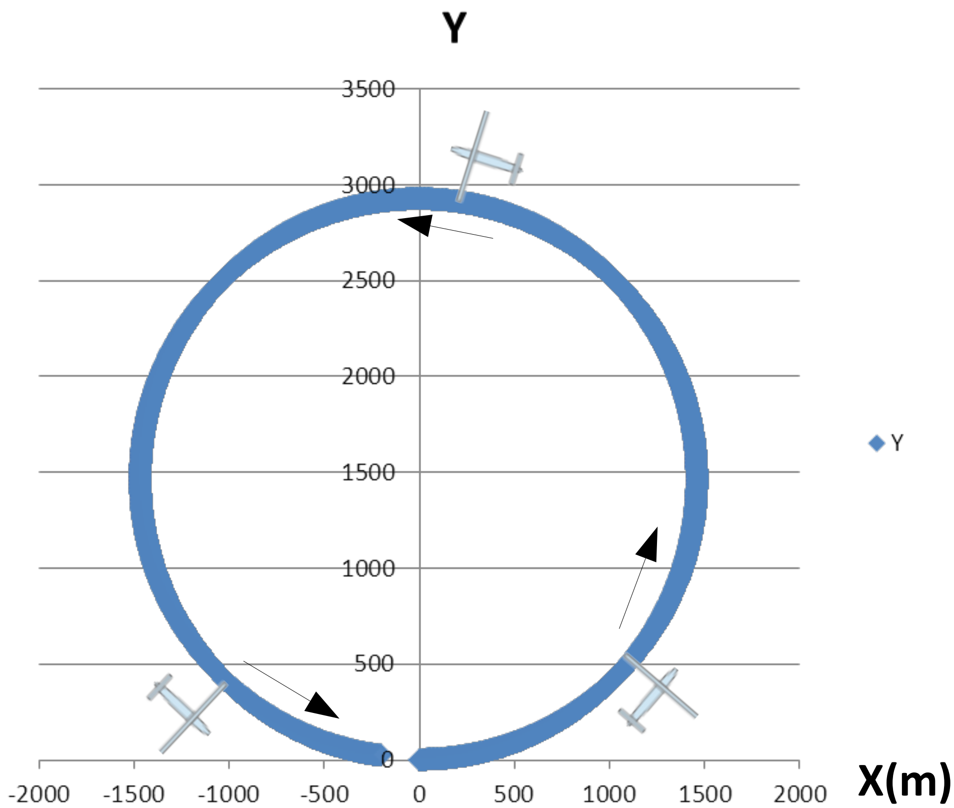
The slope of Fv at $\beta=0$ [$dFv/dt(\beta=0)$] is ~ 170 Newtons/degree or ~ 10^4 Newtons/rad.

The ratios of U1, V1 and R1 are fixed but the value of any one is not determined - it depends on initial conditions. U1 and R1 are in phase but V1 is different.

While it is not assured that the assumptions for linearization and first order exponential behavior will hold true for conditions of flight, it appears the equations might provide a solution that has the expected circular motion but with a rapidly oscillating yet decaying residual element. Numerically $\alpha = 0.1$/s (driven by the force on the vertical tail) and $\omega = 4$ rad/s (driven by the moment on the vertical tail). Note that this ω is not the angular frequency of the turn, which is $Ro = g*\sin(\Phi)/Uo$ or 0.034 rad/s for a period of 185 seconds.

B.2 Numerical

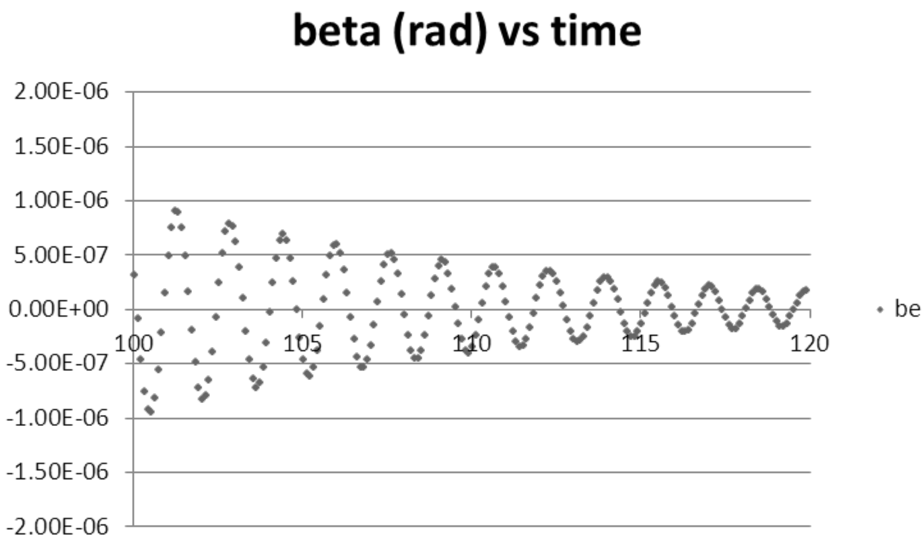
The numerical solution for the equations including Fv can be found with a simple but none too clever Euler integration method provided the time step is made small enough to avoid numerical instability. The XY trajectory that results is in the next figure. Obviously this is nearly a circle although the calculation result was not plotted, for clarity, beyond 180 seconds. If continued, the circle is very nearly retraced. The flight path is a counterclockwise near-circle. No effort was made to alter the original input parameters to get this result - it is "automatic." The radius of the circle is about 1460 meters while the nominal expected result, <u>assuming in advance a circular path</u>, is r= V^2/(g*sin(bank angle)) which yields 1440 meters. The two are very close.



Including the Fv force and moment - near-circular path for first 180 sec

This path appears to result from a dynamic balance between the horizontal lift and the force Fv with its resulting moment. Under the simple assumptions made about the very limited set of forces, it turns out the the rate of change of the yaw, R, and the longitudinal velocity U are very nearly constant (as required for a circle) but neither the relative wind sideslip angle β or the small lateral velocity V take on a constant value, as might have been expected. This is not surprising since a constant β would result in dR/dt being non-zero and constant leading to Ψ increasing like t^2 rather than just t. Rather β and V oscillate very quickly about a value of zero but with a small average value that is positive. This means that first V increases until it generates sufficient torque to advance Ψ and β which then reduces V and then the process alternates.

The following figure shows the oscillating value of β (or V) over a latter portion of the near-circular cycle. Note that the period of the very small oscillations (~1.5s) is about that suggested from the analytic evaluation above and the decay of the amplitude of β is consistent with the value of α estimated giving a decay constant of about 10 seconds.



Plot of beta(t) for a portion of the time of the trajectory above.

Even if the contribution of Fv to the force in the y direction is ignored, so long as the moment from Fv provides a non-zero N, the trajectory remains essentially the same although the time step in the integration must be be reduced to avoid numerical instability and the damping effect is lost giving higher amplitude oscillations.

Whether this is all physical reality or the consequence of simple assumptions, inadequate physical damping or even inadequate time steps has not been explored but certainly β cannot be small and constant under these assumptions.

Conclusions

It is not possible to generate an aircraft turn with nothing more than a horizontal component of lift by just banking. There must be some source of a yaw-producing moment and the vertical tail (or maybe the side of the fuselage) seems a solid candidate. The details of the moment dependence on β do not seem to matter so long as it exists, is linear near β=0 and is imposed by a finite β. This seems consistent with communications from Sadraey [3] on the need for a moment.

Of course, this is just an example of what is possible. If the full equations of motion were employed, the details of the forces and moments that produce turns are likely to be rather more complex although perhaps not essentially different.

References

1. http://www.itk.ntnu.no/fag/TTK4190/lecture_notes/2012/Aircraft%20Fossen%202011.pdf, MATHEMATICAL MODELS FOR CONTROL OF AIRCRAFT AND SATELLITES, THOR I. FOSSEN, January 2011

2. https://courses.cit.cornell.edu/mae5070/DynamicEquations.pdf

3. Thanks to Prof. Mohammad Sadraey, Daniel Webster College, private communication.

Appendix of Some Conventional (and Questionable) Wisdom on the WWW.

No mention of any moment.

1. http://www.grc.nasa.gov/WWW/K-12/airplane/turns.html

"The turn is initiated by using the ailerons or spoilers to roll, or bank, the aircraft to one side. As the aircraft is rolled, the lift vector is tilted in the direction of the roll. We can break the lift vector into two components. One component is vertical and opposed to the weight which is always directed towards the center of the earth. The other component is an unopposed side force which is in the direction of the roll, and perpendicular to the flight path. As long as the aircraft is banked, the side force is a constant, unopposed force on the aircraft. The resulting motion of the center of gravity of the aircraft is a circular arc."

2. http://www.decodedscience.org/how-does-an-airplane-turn-forces-working-to-turn-an-aircraft/11656

"An aircraft turns by banking the wings in the direction of the desired turn at a specific angle. This angle is referred to as the *bank angle*. When flying straight and level, the wings of an aircraft produce the force of lift in an upward direction, perpendicular to the wing. However, when a pilot banks the wing, this force is then divided into vertical and horizontal components. It is this horizontal component of the produced lift force, that turns the aircraft. This force, in actuality, acts as a centripetal force, trying to maintain the circular movement of the aircraft."

3. http://www.real-world-physics-problems.com/how-airplanes-fly.html

"For example, let's say a plane is to go around a horizontal turn. For a plane to make a turn its body orientation must be tilted (roll) such that the resulting aerodynamic forces enable the plane to go around a turn. Lateral forces enabling a turn are only possible by tilting the airplane such that the lift force (L) has a lateral component needed to balance the centripetal acceleration produced during the turn. . . .   By Newton's second law, the force balance for the centripetal acceleration, in the lateral direction, is given by Lsin(roll angle)=mV^2/R ... and R is the radius of the turn."